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Coexistence States for Cooperative Model with Diffusion

JIANHUA WU

Department of Mathematics
Shaanxi Normal University
Xi'an, Shaanxi 710062, P.R. China

GUANGSHENG WEI

Department of Mathematics
Xi'an Jiaotong University
Xi'an, Shaanxi 710069, P.R. China

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Abstract—In this paper, we are mainly concerned with a cooperative system with a saturating interaction term for one species. Existence of coexistence states is investigated by global bifurcation theory, and exact results on regions in parameter space which have nontrivial nonnegative steady state solutions are given. The stability of coexistence states is also studied. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we consider the cooperative system with diffusion

$$\begin{aligned}
 u_t - \Delta u &= u \left(a - u + \frac{cv}{\gamma + v} \right), & \text{in } \Omega \times (0, \infty), \\
 v_t - \Delta v &= v(b - v + du), & \text{in } \Omega \times (0, \infty), \\
 u(x, t) &= 0, & v(x, t) = 0, & \text{on } \partial\Omega \times (0, \infty), \\
 u(x, 0) &= u_0(x) \geq 0, \neq 0, & v(x, 0) &= v_0(x) \geq 0, \neq 0, & \text{in } \Omega,
 \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, Δ stands for the Laplacian operator in R^N , u and v represent the densities of two species, a and b are their growth rates which may be positive or negative, c and d are positive, and the system (1.1) is usually referred to as a cooperative system in this case.

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Both species in this model are assumed to grow logistically in absence of the other, and the interaction term for one species we consider here is often known as saturating interaction; the constant $\gamma > 0$ measures the saturation level of the species (see, for example, [1,2] for more discussion and biological background). Roughly speaking, this kind of model describes the situation whenever the species of this cooperative model are large enough, one of the species displays the saturation at first, and the rate at which the species increases is below some maximum value no matter how large the other species might be. For some kinds of species, model (1.1) is more realistic than those where the interaction term is simply assumed to be proportional to the other one.

Since we are mainly concerned with the coexistence solution of (1.1), we consider the corresponding elliptic system

$$\begin{aligned} -\Delta u &= u \left(a - u + \frac{cv}{\gamma + v} \right), & \text{in } \Omega, \\ -\Delta v &= v(b - v + du), & \text{in } \Omega, \\ u = v &= 0, & \text{on } \partial\Omega. \end{aligned} \quad (1.2)$$

For the usual cooperative system, the coexistence states have been studied in [3–5], and the two species coexist in the weak cooperative case $cd < 1$ if $a > \lambda_1$, $b > \lambda_1$ or in the strong cooperative case $cd > 1$ if $a < \lambda_1$, $b < \lambda_1$. We refer to [5] for a nice characterization of existence regions for positive solutions in the case of weak cooperation.

In this paper, our interest is to search for the coexistence state of (1.2), which refers to those steady state solutions of (1.1) having both components strictly positive in Ω . Our approach relies on various *a priori* estimates of solutions of (1.2), and other important ingredients including global bifurcation theory, fixed point index, and linear stability.

For the other studies of the cooperative model, the interested reader may be referred to [6–8].

Now we give some notation. Let $C_0(\bar{\Omega})$ be the usual Banach space of continuous functions on $\bar{\Omega}$ whose values on $\partial\Omega$ are zero, and $\|\cdot\|$ denotes the maximum norm. $C_0^+(\bar{\Omega}) = \{u \in C_0(\bar{\Omega}) : u(x) \geq 0 \text{ for } x \in \bar{\Omega}\}$, $H = [C_0^+(\bar{\Omega})]^2$, and $X = [C^1(\bar{\Omega})]^2 \cap H$. Let $\lambda_1(q)$ be the principal eigenvalue of the following problem:

$$-\Delta u + q(x)u = \lambda u, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega,$$

where $q(x)$ is a smooth function from Ω to \mathbb{R} . It is easy to see that $\lambda_1(q)$ is an increasing function of q . Let $\lambda_1(0) = \lambda_1$. The eigenfunction corresponding to λ_1 is denoted by ψ_1 . Now consider the nonlinear boundary value problem

$$-\Delta u + qu = au - u^2, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega. \quad (1.3)$$

It is well known from [9,10] that if $a \leq \lambda_1(q)$, $u = 0$ is the unique nonnegative solution of (1.3), whereas if $a > \lambda_1(q)$, (1.3) has a unique positive solution. If $q \equiv 0$ and $a > \lambda_1$, the unique positive solution is denoted by θ_a , and the mapping: $a \rightarrow \theta_a$ is strictly increasing and continuously differentiable in $(\lambda_1, +\infty)$. The corresponding initial boundary value problem is as follows:

$$\begin{aligned} u_t - \Delta u &= au - u^2, & \text{in } \Omega \times (0, +\infty), \\ u &= 0, & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) &= u_0(x) \geq 0, \neq 0, & \text{in } \Omega. \end{aligned} \quad (1.4)$$

It follows from [9,10] that $u = 0$ is globally stable if $a \leq \lambda_1$, whereas if $a > \lambda_1$, the solution $u(x, t)$ of (1.4) approaches θ_a as $t \rightarrow \infty$ uniformly for $x \in \bar{\Omega}$.

It is quite standard to show that (1.1) has a unique local solution. Global existence may be proved by the comparison theorem as usual. Next, suppose that $a + c \leq \lambda_1$. By the comparison theorem, we find

$$0 \leq u(x, t) \leq U(x, t), \quad \text{for } (x, t) \in \Omega \times [0, +\infty),$$

where $U(x, t)$ satisfies the following problem:

$$\begin{aligned} U_t - \Delta U &= (a + c)U - U^2, & \text{in } \Omega \times (0, +\infty), \\ U &= 0, & \text{on } \partial\Omega \times (0, +\infty), \\ U(x, 0) &= u_0(x) \geq 0, \neq 0, & \text{in } \Omega. \end{aligned}$$

Since $\lim_{t \rightarrow \infty} U = 0$ for $a + c \leq \lambda_1$, we have $\lim_{t \rightarrow \infty} u = 0$. Thus, $L_\omega(u_0, v_0)$, the ω -limit set of the mapping $S(t) : (u_0, v_0) \rightarrow (u, v)$, lies in $\{0\} \times C_0^+$, and the solution (u, v) of (1.1), corresponding to $u_0 = 0, v_0 \geq 0, \neq 0$, always satisfies $u = 0$ and

$$\begin{aligned} v_t - \Delta v &= bv - v^2, & \text{in } \Omega \times (0, \infty), \\ v &= 0, & \text{on } \partial\Omega \times (0, \infty), \\ v(x, 0) &= v_0(x) \geq 0, \neq 0, & \text{in } \Omega. \end{aligned}$$

Hence $\lim_{t \rightarrow \infty} v = 0$ for $b \leq \lambda_1$ and $\lim_{t \rightarrow \infty} v = \theta_b$ for $b > \lambda_1$. For completeness, we state this preliminary result as follows.

THEOREM 1. *Suppose $(u_0, v_0) \in H$. Then (1.1) has a unique pair of nonnegative solution (u, v) which exists for all $t > 0$. Moreover, if $a + c \leq \lambda_1$, the solution (u, v) of (1.1) tends to $(0, 0)$ as $t \rightarrow +\infty$ for $b \leq \lambda_1$, and to $(0, \theta_b)$ as $t \rightarrow +\infty$ for $b > \lambda_1$.*

REMARK 1. Even though (1.1) is a cooperative system, as long as the response function is controlled properly, the blow-up never occurs, which is somewhat different from the usual cooperative system. Furthermore, if $a + c \leq \lambda_1$, it follows from Theorem 1 that there exists no coexistence state of (1.2).

Throughout this paper, the global bifurcation follows the same definition as [11] in $R \times X$. Now we are ready to present our main result. First, if a is a little larger, our next result shows that there exists coexistence state by increasing b suitably. More precisely, we have the following theorem.

THEOREM 2. *Suppose $\lambda_1 - c < a < \lambda_1$. Then the global bifurcation C of positive solutions of (1.2) occurs at $(\beta, 0, \theta_\beta)$, which tends to ∞ in $R \times X$ and corresponds to the coexistence state of (1.2), where $\beta \in (\lambda_1, +\infty)$ is uniquely determined by $a = \lambda_1(-c\theta_\beta/(\gamma + \theta_\beta))$.*

REMARK 2. From the proof of the above theorem in Section 2, it can be easily seen that $\lim_{b \rightarrow \infty} v/b = 1$ almost everywhere in Ω for $(b, u, v) \in C$.

If a increases further, the coexistence state can occur even in the case of a little smaller b . We give the result precisely.

THEOREM 3. *Suppose $a > \lambda_1$. Then the global bifurcation C of positive solutions of (1.2) emanates from $(b', \theta_a, 0)$ and tends to ∞ in $R \times X$, which corresponds to the coexistence state of (1.2), where $b' = \lambda_1(-d\theta_a)$.*

REMARK 3. For simplicity, we have assumed that the rates at which both species u and v diffuse equal one, and it is easy to check that Theorems 1–3 hold with minor modification even in the case of different diffusion constants.

Next, we discuss the stability of the coexistence state in Theorem 3 in the neighbourhood of the bifurcation point $(b', \theta_a, 0)$. Let $\mu = \int_\Omega \chi_1^2(\chi_1 - dw_1) dx$, where (w_1, χ_1) is given in Section 3. If $a > \lambda_1$ and $\mu < 0$, then we can conclude from Section 4 below that $b(s) < b'$ for $0 < s \ll 1$, where $b(s)$ is given in Section 3. Denote $\underline{b} = \inf\{b : (b, u, v) \in C\}$. Then we find $\underline{b} < b'$. It follows from Theorem 3 that the global bifurcation C tends to $+\infty$ in $R \times X$ as $b \rightarrow +\infty$. Thus we have shown that, in the case of $\mu < 0$, there exists a set $\Upsilon \subset [\underline{b}, b')$, $\Upsilon \neq \emptyset$, such that (1.2) has at least two coexistence states for $b \in \Upsilon$. Furthermore, we have the following theorem.

THEOREM 4. Suppose $a > \lambda_1$ and $\mu < 0$. Then the coexistence state in the neighbourhood of $(b', \theta_a, 0)$ in Theorem 3 is unstable, and $\{b : (b, u, v) \in C\} = [b, \infty)$.

PROPOSITION 1. Suppose $a > \lambda_1$ and $c \geq \gamma(1 + 2/d)$. Then $\mu < 0$. Hence, the conclusion of Theorem 4 holds.

REMARK 4. For fixed γ , if c is suitably large or $c > \gamma$ is suitably given but d is suitably large, then the condition of Proposition 1 can hold.

However, if c is sufficiently small, we have the following proposition.

PROPOSITION 2. Suppose that $a > \lambda_1$. If $b < b'$ and c is sufficiently small, then $(\theta_a, 0)$ is globally attracting for the problem (1.1). Hence, there is no coexistence state in this case; if $b > b'$ and c is sufficiently small, then there exists a unique coexistence state of (1.2), which is linearly stable and lies on the global bifurcation C . Moreover, $\{b : (b, u, v) \in C\} = (b', +\infty)$.

This paper is organized as follows. In Section 2, we give the proof of Theorem 2, the global bifurcation theory, and the fixed point index being used. In Section 3, the coexistence state is investigated in the case of $a > \lambda_1$. Finally, in Section 4, we give the stability analysis of the coexistence state obtained in Section 3.

2. THE EXISTENCE OF COEXISTENCE STATE (I)

In this section, we give the proof of Theorem 2. For this purpose, it is necessary to obtain some *a priori* information about solutions of (1.2). First we have the following lemma.

LEMMA 1. If (u, v) is a nontrivial nonnegative solution of (1.2), and $u \not\equiv 0$, then $a + c > \lambda_1$.

PROOF. It follows from the maximum principle that $u > 0$ in Ω . Multiplying the first equation of (1.2) by u , and integrating over Ω , we find

$$\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} u^2 \left(a - u + \frac{cv}{\gamma + v} \right) dx < \int_{\Omega} u^2 (a + c) dx.$$

By virtue of the Poincaré inequality, we have $a + c > \lambda_1$.

LEMMA 2. Suppose (u, v) is the nontrivial nonnegative solution of (1.2) such that $u \not\equiv 0$, $v \not\equiv 0$. If $\lambda_1 - c < a \leq \lambda_1$, then $0 < u \leq \theta_{(a+c)} \leq a + c$; and $u \geq \theta_a$ whenever $a > \lambda_1$; furthermore, if $b + d(a + c) > \lambda_1$, we have $0 < v \leq \theta_{(b+d(a+c))}$, and $v \geq \theta_b$ whenever $b > \lambda_1$.

PROOF. By the maximum principle, we have $u > 0$ and $v > 0$ in Ω . It is easily seen that u is a subsolution of the following problem:

$$-\Delta \theta = \theta(a + c - \theta), \quad \text{in } \Omega, \quad \theta = 0, \quad \text{on } \partial\Omega,$$

and there exists a sufficiently large constant supersolution for θ . By the uniqueness of $\theta_{(a+c)}$, we have $u \leq \theta_{(a+c)}$ in Ω . The remaining part of the lemma can be shown similarly, and thus we omit it here.

Let $u = U$, $v = V + \theta_b$. It is easy to check that (U, V) satisfies

$$\begin{aligned} -\Delta U &= aU + \frac{c\theta_b}{\gamma + \theta_b}U + f_1(x, U, V), \\ -\Delta V &= bV - 2\theta_b V + d\theta_b U + f_2(x, U, V), \end{aligned}$$

where

$$\begin{aligned} f_1(x, U, V) &= -U^2 + \left(\frac{c(V + \theta_b)}{\gamma + (V + \theta_b)} - \frac{c\theta_b}{\gamma + \theta_b} \right) U, \\ f_2(x, U, V) &= -V^2 + dUV. \end{aligned}$$

Let K be the inverse of $-\Delta$ with Dirichlet boundary condition. Then we find

$$\begin{aligned} U &= aKU + cK\left(\frac{\theta_b U}{\gamma + \theta_b}\right) + KF_1(U, V), \\ V &= bKV - 2K(\theta_b V) + dK(\theta_b U) + KF_2(U, V), \end{aligned} \quad (2.1)$$

where $F_i(U, V)$ is defined by $F_i(U, V)(x) = f_i(x, U, V)$.

We define T by $T(b, U, V) = (aKU + cK(\theta_b U/(\gamma + \theta_b)) + KF_1(U, V), bKV - 2K(\theta_b V) + dK(\theta_b U) + KF_2(U, V))$. Let $G = I - T$. Then $G(b, U, V) = 0$ with $U \geq 0$ and $V \geq 0$ if and only if $(b, U, V + \theta_b)$ is a nonnegative solution of (1.2).

Let $L(b) = -\Delta - b + 2\theta_b$. Then we have the following lemma.

LEMMA 3. Suppose $b > \lambda_1$. Then all the eigenvalues of $L(b)$ are positive.

By the definition of θ_b and the monotonicity of the principal eigenvalue, Lemma 3 can be proved, and the proof is omitted here.

It is easy to see that $\lambda_1(-c\theta_b/(\gamma + \theta_b))$ is a decreasing function with respect to b which equals λ_1 when $b = \lambda_1$ and tends to $\lambda_1 - c$ as $b \rightarrow \infty$. If $\lambda_1 - c < a < \lambda_1$, then there is a unique value $\beta > \lambda_1$ such that $a = \lambda_1(-c\theta_\beta/(\gamma + \theta_\beta))$.

LEMMA 4. Suppose $\lambda_1 - c < a < \lambda_1$. Then $(\beta, 0, \theta_\beta)$ is a bifurcation point for (2.1) and there exist positive solutions of (2.1) in the neighbourhood of $(\beta, 0, \theta_\beta)$.

PROOF. It is easy to see that G is a C^1 function. The Frechet derivative of G with respect to (U, V) at $(\beta, 0, 0)$ is denoted by L_1 . Then

$$\begin{aligned} L_1 \cdot (w, \chi) &= DG_{(U, V)}(\beta, 0, 0) \cdot (w, \chi) \\ &= \left(w - aKw - cK\left(\frac{\theta_\beta w}{\gamma + \theta_\beta}\right), \chi - \beta K\chi + 2K(\theta_\beta \chi) - dK(\theta_\beta w) \right). \end{aligned}$$

It follows that $N(L_1) = \text{span}\{(w_0, \chi_0)\}$, where w_0 is a nonnegative principal eigenfunction corresponding to the principal eigenvalue problem

$$-\Delta w - \frac{c\theta_\beta}{\gamma + \theta_\beta} w = aw, \quad \text{in } \Omega, \quad w = 0, \quad \text{on } \partial\Omega,$$

and $\chi_0 = L^{-1}(\beta)(d\theta_\beta w_0)$. Thus, $\dim N(L_1) = 1$. It is easy to check that $N(L_1^*) = \text{span}\{(w_0, 0)\}$, where the adjoint operator of L_1 is denoted by L_1^* . By the Fredholm alternative, we have $\text{codim } R(L_1) = 1$.

By differentiating L_1 with respect to b , we have

$$\begin{aligned} L_2(w_0, \chi_0) &\triangleq D^2 G_{b, (U, V)}(\beta, 0, 0) \cdot (w_0, \chi_0) \\ &= \left(-c\gamma K\left(\frac{\theta'_\beta w_0}{(\gamma + \theta_\beta)^2}\right), -K\chi_0 + 2K(\theta'_\beta \chi_0) - dK(\theta'_\beta w_0) \right), \end{aligned}$$

where L_2 is the Frechet derivative of second order of G at $(\beta, 0, 0)$ and θ'_b is the derivative of θ_b with respect to b .

Suppose that $L_2(w_0, \chi_0) \in R(L_1)$. Then there exists w such that

$$w - aKw - cK\left(\frac{\theta_\beta w}{\gamma + \theta_\beta}\right) = -c\gamma K\left(\frac{\theta'_\beta w_0}{(\gamma + \theta_\beta)^2}\right).$$

Thus, we have

$$-\Delta w = aw + \frac{c\theta_\beta}{\gamma + \theta_\beta} w - \frac{c\gamma\theta'_\beta w_0}{(\gamma + \theta_\beta)^2}.$$

Multiplying by w_0 , integrating over Ω , and noting the definition of w_0 , we get

$$0 = -c\gamma \int_{\Omega} \frac{\theta'_\beta w_0^2}{(\gamma + \theta_\beta)^2} dx.$$

Since $\theta'_\beta > 0$, this is impossible, which leads to $L_2(w_0, \chi_0) \notin R(L_1)$. Thus, Theorem 1.7 of [12] can be applied, and there exists a curve of nontrivial nonnegative solution of (2.1) bifurcating at $(\beta, 0, \theta_\beta)$, and $(b(s), U(s), V(s))(|s| < \delta)$ is the unique nontrivial solution branch in the neighbourhood of $(\beta, 0, 0)$, where $U(s) = s(w_0 + \phi(s))$, $V(s) = s(\chi_0 + \psi(s))$ and $\phi(s), \psi(s) \in C^1(-\delta, +\delta)$, $\phi(0) = \psi(0) = 0$. By the definition of χ_0 , it follows from Lemma 3 and the generalized maximum principle that $\chi_0 > 0$. Thus, $(u(s), v(s))$ with $0 < s < \delta$ is the coexistence state of (1.2), where $u(s) = U(s)$, $v(s) = \theta_b + V(s)$.

Let $T : R \times \tilde{X} \rightarrow \tilde{X}$ be a compact continuously differentiable operator such that $T(b, 0) = 0$. Suppose we can write T as $T(b, u) = K(b)u + R(b, u)$, where $K(b)$ is a linear compact operator and the Frechet derivative $R_u(b, 0) = 0$. We investigate the bifurcation solution for $u = T(b, u)$ by treating b as a bifurcation parameter. If x_0 is an isolated fixed point of T , we define the index of T at x_0 as $i(T, x_0) = \deg(I - T, B, x_0)$, where B is a ball with centre at x_0 such that x_0 is the only fixed point of T in B . If $I - T'(x_0)$ is invertible, then x_0 is an isolated fixed point of T and

$$i(T, x_0) = \deg(I - T, B, x_0) = \deg(I - T'(x_0), \hat{B}, 0).$$

If $x_0 = 0$, then it is well known that the Leray-Schauder degree $\deg(I - K(b), \hat{B}, 0) = (-1)^\sigma$, where $\sigma =$ sum of the algebraic multiplicities of the eigenvalue of K which is greater than one.

We state the result that we use next on the global bifurcation, which is essentially due to Crandall and Rabinowitz [11].

LEMMA 5. Let b_0 be such that $I - K(b)$ is invertible if $0 < |b - b_0| < \epsilon$ for some $\epsilon > 0$. Suppose $i(T(b, \cdot), 0)$ is constant on $(b_0 - \epsilon, b_0)$ and on $(b_0, b_0 + \epsilon)$ such that if $b_0 - \epsilon < b_1 < b_0 < b_2 < b_0 + \epsilon$, then $i(T(b_1, \cdot), 0) \neq i(T(b_2, \cdot), 0)$. Then there exists a continuum C in the b - u plane of solutions of $u = T(b, u)$ such that one of the following alternatives holds:

- (i) C joins $(b_0, 0)$ with $(\hat{b}, 0)$ where $I - K(\hat{b})$ is not invertible and $\hat{b} \neq b_0$;
- (ii) C joins $(b_0, 0)$ to ∞ in $R \times \tilde{X}$.

Now we are going to complete the proof of Theorem 2. We divide the proof into several steps.

STEP 1. Let $T'(b)$ be defined by

$$\begin{aligned} T'(b) \cdot (w, \chi) &= DT_{(U,V)}(b, 0, 0) \cdot (w, \chi) \\ &= \left(aKw + cK\left(\frac{\theta_b w}{\gamma + \theta_b}\right), bK\chi - 2K(\theta_b \chi) + dK(\theta_b w) \right). \end{aligned}$$

It is easy to see that $\mu \geq 1$ is an eigenvalue of $T'(b)$ if and only if a is the eigenvalue of the following problem:

$$-\mu \Delta w - \frac{c\theta_b}{\gamma + \theta_b} w = aw, \quad \text{in } \Omega, \quad w = 0, \quad \text{on } \partial\Omega. \quad (2.2)$$

Suppose $b < \beta$ and $\mu \geq 1$ is an eigenvalue of $T'(b)$. Then a is an eigenvalue of (2.2). The i^{th} eigenvalue of (2.2) is denoted by $\lambda_i(\mu, -c\theta_b/(\gamma + \theta_b))$ ($i = 1, 2, \dots$). It follows from the monotonicity of the principal eigenvalue that

$$a = \lambda_1\left(-\frac{c\theta_\beta}{\gamma + \theta_\beta}\right) < \lambda_1\left(-\frac{c\theta_b}{\gamma + \theta_b}\right) \leq \lambda_1\left(\mu, -\frac{c\theta_b}{\gamma + \theta_b}\right).$$

So $w \equiv 0$, which gives a contradiction. Thus, the multiplicity σ of the eigenvalue of $T'(b)$ which is greater than one is zero. Therefore, $i(T'(b)) = 1$.

Suppose $\beta < b < \beta + \epsilon$ such that $\lambda_1(-c\theta_b/(\gamma + \theta_b)) < a < \lambda_2(-c\theta_b/(\gamma + \theta_b))$ and $\mu \geq 1$ is an eigenvalue of $T'(b)$. Then a must be one of the eigenvalues of (2.2). Since for $i \geq 2$, $\lambda_i(\mu, -c\theta_b/(\gamma + \theta_b)) \geq \lambda_i(-c\theta_b/(\gamma + \theta_b)) \geq \lambda_2(-c\theta_b/(\gamma + \theta_b)) > a$, and $\lambda_1(\mu, -c\theta_b/(\gamma + \theta_b))$ is increasing with respect to μ and equals $\lambda_1(-c\theta_b/(\gamma + \theta_b))$ when $\mu = 1$ tends to ∞ as $\mu \rightarrow +\infty$, then there exists a unique $\mu_1 > 1$ such that $a = \lambda_1(\mu_1, -c\theta_b/(\gamma + \theta_b))$. Let w^0 be the corresponding principal eigenfunction

$$-\mu_1 \Delta w = aw + \frac{c\theta_b w}{\gamma + \theta_b}, \quad \text{in } \Omega, \quad w = 0, \quad \text{on } \partial\Omega.$$

It follows from Lemma 3 and $\mu_1 > 1$ that all the eigenvalues of the operator $L(\mu_1, b) = -\mu_1 \Delta - b + 2\theta_b$ are positive. Thus, there exists a unique solution of the following problem, denoted by $\chi^0 = L^{-1}(\mu_1, b)(d\theta_b w^0)$:

$$-\mu_1 \Delta \chi = b\chi - 2\theta_b \chi + d\theta_b w^0, \quad \text{in } \Omega, \quad \chi = 0, \quad \text{on } \partial\Omega.$$

Hence, we have $N(\mu_1 I - T'(b)) = \text{spans}\{(w^0, \chi^0)\}$ and $\dim N(\mu_1 I - T'(b)) = 1$. It remains to prove that $R(\mu_1 I - T'(b)) \cap N(\mu_1 I - T'(b)) = 0$. Suppose that the above assertion is false. Then we may assume that $(w^0, \chi^0) \in R(\mu_1 I - T'(b))$. Hence, there exists $(w, \chi) \in X$ such that

$$\mu_1 \Delta w + aw + \frac{c\theta_b w}{\gamma + \theta_b} = \Delta w^0, \quad \text{in } \Omega, \quad w = 0, \quad \text{on } \partial\Omega.$$

Multiplying both sides of the equation by w^0 , integrating over Ω , and using Green's formula, we have

$$\begin{aligned} \int_{\Omega} w^0 \Delta w^0 dx &= \int_{\Omega} \left(\mu_1 \Delta w + aw + \frac{c\theta_b w}{\gamma + \theta_b} \right) w^0 \\ &= \int_{\Omega} \left(\mu_1 \Delta w^0 + aw^0 + \frac{c\theta_b w^0}{\gamma + \theta_b} \right) w = 0, \end{aligned}$$

which gives a contradiction to the definition of w^0 . Thus, the multiplicity of μ_1 is one in this case, which gives $i(T'(b)) = -1$ for $\beta < b < \beta + \epsilon$.

According to Lemma 5, we obtain that there exists a continuum C_0 of solution of $(U, V) = T(b, U, V)$, which either joins with $(\hat{b}, 0, 0)$, where $I - T'(\hat{b})$ is not invertible and $\hat{b} \neq \beta$, or joins to ∞ in $R \times X$. Close to the bifurcation point the continuum C_0 consists of the solution branch given by Lemma 4. Let $C_1 = C_0 - \{(b(s), s(w_0 + \phi(s)), s(\chi_0 + \psi(s))) : -\delta < s < 0\}$. We can proceed as Rabinowitz [11] to show that C_1 either satisfies one of the same alternatives as C_0 or contains a pair of points of the form (b, U, V) and $(b, -U, -V)$ where $(U, V) \neq (0, 0)$. Set $C = \{(b, u, v) : u = U, v = V + \theta_b \text{ and } (b, U, V) \in C_1\}$. Then (b, u, v) is a solution of (1.2) whenever $(b, u, v) \in C$.

STEP 2. Let $P_1 = \{u \in C_0^1(\bar{\Omega}) : u(x) > 0 \text{ for } x \in \Omega, \frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega\}$ and

$$P = \{(b, u, v) : u, v \in P_1 \text{ for } x \in \Omega\}.$$

Clearly, $C \subset P$ in the neighbourhood of the bifurcation point $(\beta, 0, \theta_\beta)$.

Next we claim that the following assertions hold.

(A₁) $C - \{(\beta, 0, \theta_\beta)\} \subset P$.

(A₂) $\{b : (b, u, v) \in C\} = [\bar{\beta}, +\infty)$ for some $\bar{\beta} \geq \beta - d(a + c)$.

REMARK 5. It is easy to see that Theorem 2 follows from assertion (A₂).

PROOF OF (A₁). Suppose that $C - \{(\beta, 0, \theta_\beta)\}$ is not contained in P . Then there exists $(\hat{b}, \hat{u}, \hat{v}) \in \{C - (\beta, 0, \theta_\beta)\} \cap \partial P$ which is the limit of a sequence of points $\{(b_n, u_n, v_n)\}$ in $C \cap P$. As $(\hat{u}, \hat{v}) \in \partial P$, either $\hat{u} \in \partial P_1$ or $\hat{v} \in \partial P_1$.

Suppose $\hat{u} \in \partial P_1$. Then $\hat{u} \geq 0$ for $x \in \Omega$ and either $\hat{u} = 0$ for some $x \in \Omega$ or $\frac{\partial \hat{u}}{\partial n} = 0$ for some $x \in \partial\Omega$. Since \hat{u} satisfies

$$-\Delta \hat{u} = \hat{u} \left(a - \hat{u} + \frac{c\hat{v}}{\gamma + \hat{v}} \right), \quad \text{in } \Omega, \quad \hat{u} = 0, \quad \text{on } \partial\Omega,$$

it follows from the maximum principle that $\hat{u} \equiv 0$. Similar argument shows that if $\hat{v} \in \partial P_1$, then $\hat{v} = 0$. Thus, $\hat{u} = 0$ or $\hat{v} = 0$.

Suppose $(\hat{u}, \hat{v}) = (0, 0)$. The only nontrivial nonnegative solution branch bifurcating from $\{(b, 0, 0) : b \in R\}$ is $\{(b, 0, \theta_b) : b > \lambda_1\}$. Since $(b_n, u_n, v_n) \in C \cap P$, which leads to a contradiction, thus either \hat{u} or \hat{v} is zero.

Suppose $\hat{v} \equiv 0$ and $\hat{u} > 0$. Then \hat{u} satisfies

$$-\Delta \hat{u} = \hat{u}(a - \hat{u}), \quad \text{in } \Omega, \quad \hat{u} = 0, \quad \text{on } \partial\Omega.$$

Since $a < \lambda_1$, we get $\hat{u} \equiv 0$, which gives a contradiction.

Suppose $\hat{u} \equiv 0$ and $\hat{v} > 0$. It is easy to see that $(\hat{b}, \hat{u}, \hat{v})$ lies on $\{(b, 0, \theta_b) : b > \lambda_1\}$, and from which bifurcates a nontrivial nonnegative solution. Let $\bar{u}_n = u_n / \|u_n\|$. Then \bar{u}_n satisfies

$$-\Delta \bar{u}_n = \bar{u}_n \left(a - u_n + \frac{cv_n}{\gamma + v_n} \right), \quad \text{in } \Omega, \quad \bar{u}_n = 0, \quad \text{on } \partial\Omega.$$

Since (\bar{u}_n, v_n) is bounded as displayed in Lemma 2, and the above equation can be also regarded as a fixed point of a compact operator, we can extract a convergent subsequence of (\bar{u}_n, v_n) , again labeled by n , such that $\bar{u}_n \rightarrow \bar{u}$, $v_n \rightarrow \theta_{\hat{b}}$. Moreover, $\bar{u} \geq 0, \neq 0$ because of $\|\bar{u}\| = 1$, and satisfies

$$-\Delta \bar{u} = \bar{u} \left(a + \frac{c\theta_{\hat{b}}}{\gamma + \theta_{\hat{b}}} \right), \quad \text{in } \Omega, \quad \bar{u} = 0, \quad \text{on } \partial\Omega.$$

It follows from the maximum principle that $\bar{u} > 0$. Thus $a = \lambda_1(-c\theta_{\hat{b}}/(\gamma + \theta_{\hat{b}}))$, which gives $\hat{b} = \beta$, a contradiction. In conclusion, Assertion (A₁) is proved.

PROOF OF (A₂). It is easy to see by (A₁) that C joins $\{(\beta, 0, \theta_\beta)\}$ to ∞ in $R \times X$. Suppose $(b, u, v) \in C - \{(\beta, 0, \theta_\beta)\}$. Then $(u, v) > (0, 0)$ in Ω , and satisfies

$$\begin{aligned} -\Delta u &= u \left(a - u + \frac{cv}{\gamma + v} \right), & \text{in } \Omega, \\ -\Delta v &= v(b - v + du), & \text{in } \Omega, \\ u = v &= 0, & \text{on } \partial\Omega. \end{aligned}$$

It follows that $b > \lambda_1(-du)$ and $a > \lambda_1(-cv/(\gamma + v))$. By Lemma 2, we find $b > \lambda_1(-d\theta_{(a+c)})$. Hence, $b > \lambda_1 - d(a + c)$. Therefore, $0 < v \leq \theta_{(b+d(a+c))}$, which leads to

$$a > \lambda_1 \left(-\frac{cv}{\gamma + v} \right) \geq \lambda_1 \left(-\frac{c\theta_{(b+d(a+c))}}{\gamma + \theta_{(b+d(a+c))}} \right).$$

Thus, $b \geq \beta - d(a + c)$ because of $a = \lambda_1(-c\theta_\beta/(\gamma + \theta_\beta))$.

If $\{b : (b, u, v) \in C\}$ is bounded in R , it follows from Lemma 2 and L^p estimate that there exists a constant M such that $\|u\|, \|v\| \leq M$. Hence, C cannot join the point $\{(\beta, 0, \theta_\beta)\}$ to ∞ in $R \times X$, a contradiction. Since $\{b : (b, u, v) \in C\}$ is connected, so it must be $[\bar{\beta}, \infty)$ for some $\bar{\beta} \geq \beta - d(a + c)$. Moreover, by combination of Lemma 2 and Proposition 6.5 (see [13]), we can find that $\lim_{b \rightarrow \infty} v/b = 1$ almost everywhere in Ω .

3. THE EXISTENCE OF COEXISTENCE STATE (II)

In this section, we give the proof of Theorem 3. Given $a > \lambda_1$, we regard b as bifurcation parameter once again. System (1.2) has the trivial branch $\{(b, 0, 0) : b \in R\}$, the semitrivial branch $\{(b, \theta_a, 0) : b \in R\}$, and $\{(b, 0, \theta_b) : b > \lambda_1\}$.

Let $u = \tilde{u} + \theta_a$, $v = \tilde{v}$. For the sake of simplicity, we still denote \tilde{u} and \tilde{v} by u and v . Then we have

$$\begin{aligned} -\Delta u &= (a - 2\theta_a)u + \frac{c}{\gamma}\theta_a v + f_1(u, v), & \text{in } \Omega, \\ -\Delta v &= (b + d\theta_a)v + f_2(u, v), & \text{in } \Omega, \\ u = v &= 0, & \text{on } \partial\Omega, \end{aligned}$$

where

$$\begin{aligned} f_1(u, v) &= -u^2 + \frac{c(u + \theta_a)v}{\gamma + v} - \frac{c}{\gamma}\theta_a v, \\ f_2(u, v) &= -v^2 + duv. \end{aligned}$$

As usual, we define T by $T(b, u, v) = (aKu - 2K(\theta_a u) + (c/\gamma)K(\theta_a v) + Kf_1(u, v), bKv + dK(\theta_a v) + Kf_2(u, v))$. Let $G = I - T$. Then $G(b, u, v) = 0$ with $u \geq 0$ and $v \geq 0$ if and only if $(b, u + \theta_a, v)$ is the solution of (1.2). The Frechet derivative of G with respect to (u, v) at $(b, 0, 0)$ is given by

$$G_{(u,v)}(b, 0, 0) \cdot (w, \chi) = \left(w - aKw + 2K(\theta_a w) - \frac{c}{\gamma}K(\theta_a \chi), \chi - bK\chi - dK(\theta_a \chi) \right).$$

Let $b' = \lambda_1(-d\theta_a)$ and $\chi_1 > 0$ denote the corresponding principal eigenvalue and eigenfunction of the following problem, normalized χ_1 as $\int_{\Omega} \chi_1^2 dx = 1$:

$$-\Delta \chi = b\chi + d\theta_a \chi, \quad \text{in } \Omega, \quad \chi = 0, \quad \text{on } \partial\Omega.$$

Denote $w_1 = L^{-1}(a)((c/\gamma)\theta_a \chi_1)$. It follows from the generalized maximum principle that $w_1 > 0$. Next we can proceed as in Section 2 to obtain that $(b', \theta_a, 0)$ is a bifurcation point, and in the neighbourhood of $(b', \theta_a, 0)$, $(b(s), u(s), v(s))$ with $0 < s \ll 1$ is the only coexistence state of (1.2), where $u(s) = \theta_a + s(w_1 + \Phi(s))$, $v(s) = s(\chi_1 + \Psi(s))$ and $b(s)$, $\Phi(s)$, $\Psi(s) \in C^1$, $b(0) = b'$, $\Phi(0) = \Psi(0) = 0$. Furthermore, there exists a continuum C of (1.2) in $R \times X$ bifurcating from $(b', \theta_a, 0)$ and satisfying one of the following three alternatives:

- (i) C contains points of the form $(b, \theta_a + u, v)$ and $(b, \theta_a - u, -v)$ with $u \neq 0$, $v \neq 0$;
- (ii) C joins with a bifurcation point of the form $(\hat{b}, \theta_a, 0)$, where $\hat{b} \neq b'$;
- (iii) C joins $(b', \theta_a, 0)$ to ∞ in $R \times X$.

We are in a position to complete the proof of Theorem 3, which is established by the following lemma.

LEMMA 6. $C - \{(b', \theta_a, 0)\}$ joins $\{(b', \theta_a, 0)\}$ to ∞ in P .

PROOF. First we show that $C - \{(b', \theta_a, 0)\} \subset P$.

If the assertion is false, then there exists $(\hat{b}, \hat{u}, \hat{v}) \in [C - \{(b', \theta_a, 0)\}] \cap \partial P$, which is the limit of a sequence $\{(b_n, u_n, v_n)\} \subset C \cap P$. Similar argument as in the proceeding section shows that $\hat{u} = 0$ or $\hat{v} = 0$.

If $(\hat{u}, \hat{v}) = 0$, a contradiction is given as before.

If $\hat{u} = 0$, $\hat{v} > 0$, we can proceed as the proof of (A_1) in Section 2 to obtain that there exists $\bar{u} > 0$ in Ω such that

$$\begin{aligned} -\Delta \bar{u} &= \bar{u} \left(a + \frac{c\hat{v}}{\gamma + \hat{v}} \right), \\ -\Delta \hat{v} &= \hat{v} (\hat{b} - \hat{v}). \end{aligned}$$

Since $\hat{v} > 0$, we get $\hat{b} > \lambda_1$ and $\hat{v} = \theta_{\hat{b}}$. It follows from the maximum principle that $\bar{u} > 0$ in Ω . By application of the Krein-Rutman theorem, we have $a = \lambda_1(-c\theta_{\hat{b}}/(\gamma + \theta_{\hat{b}}))$, a contradiction to $a > \lambda_1$.

If $\hat{u} > 0$ and $\hat{v} = 0$, since $(b_n, u_n, v_n) \rightarrow (\hat{b}, \hat{u}, 0)$ and satisfies

$$\begin{aligned} -\Delta u_n &= u_n \left(a - u_n + \frac{cv_n}{\gamma + v_n} \right), & \text{in } \Omega, \\ -\Delta v_n &= v_n(b_n - v_n + du_n), & \text{in } \Omega, \\ u_n &= v_n = 0, & \text{on } \partial\Omega, \end{aligned}$$

then $V_n = v_n/\|v_n\|$ satisfies

$$-\Delta V_n = V_n(b_n - v_n + du_n).$$

We can proceed as above to prove that $V_n \rightarrow v$, where $v \geq 0$, $\neq 0$, and (\hat{u}, v) is the solution of the following problem:

$$\begin{aligned} -\Delta \hat{u} &= \hat{u}(a - \hat{u}), \\ -\Delta v &= v(\hat{b} + d\hat{u}). \end{aligned}$$

Since $a > \lambda_1$ and $\hat{u} > 0$, we have $\hat{u} = \theta_a$. It follows from the maximum principle that $v > 0$. Hence $\hat{b} = \lambda_1(-d\theta_a) = b'$, a contradiction.

In conclusion, the assertion given above is proved. Therefore, (i),(ii) are impossible, and the only remaining possibility is that C joins $(b', \theta_a, 0)$ to ∞ in $R \times X$. Thus, if $(b, u, v) \in C - (b', \theta_a, 0)$, then $(u, v) > (0, 0)$ and satisfies

$$-\Delta v = v(b - v + du), \quad \text{in } \Omega, \quad v = 0, \quad \text{on } \partial\Omega.$$

So $b > \lambda_1(-du) \geq \lambda_1(-\theta_{(a+c)}) \geq \lambda_1 - d(a+c)$ and $0 < v \leq \theta_{(b+d(a+c))}$. Suppose that $\{b : (b, u, v) \in C\}$ is bounded in R . Since $\theta_a \leq u \leq \theta_{(a+c)}$, from L^p estimate and the Sobolev embedding theorem, it follows that $\|u\| \leq M$, $\|v\| \leq M$ for some constant $M > 0$, a contradiction. Therefore, we obtain that C tends to ∞ in $R \times X$ only by increasing b unboundedly.

4. THE STABILITY OF THE COEXISTENCE STATE (II)

In this final section, we discuss the stability of the coexistence state in Theorem 3 in the neighbourhood of $(b', \theta_a, 0)$. Let $X_1 = [C^2(\bar{\Omega})]^2 \cap X$, $Y_1 = [C(\bar{\Omega})]^2$, $i : X_1 \rightarrow Y_1$ be the inclusion mapping, and $L(b', \theta_a, 0)$ be the linearized operator of (1.2) at $(b', \theta_a, 0)$. Then the proof of Theorem 4 can be completed by the following Lemmas 7–11.

LEMMA 7. *0 is an i -simple [12] eigenvalue of $L(b', \theta_a, 0)$, and all the other eigenvalues of $L(b', \theta_a, 0)$ lie in the right half complex plane.*

PROOF. It follows from Section 3 that 0 is the eigenvalue of $L(b', \theta_a, 0)$, and $N(L(b', \theta_a, 0)) = \text{spans}\{(w_1, \chi_1)\}$.

Let $L^*(b', \theta_a, 0)$ be the adjoint operator of $L(b', \theta_a, 0)$. Then we have $N(L^*(b', \theta_a, 0)) = \text{spans}\{(0, \chi_1)\}$. Fredholm alternative gives $R(L(b', \theta_a, 0))$, the range of $L(b', \theta_a, 0)$, as follows:

$$R(L(b', \theta_a, 0)) = \left\{ (h_1, h_2) \in Y_1 : \int_{\Omega} h_2(x) \chi_1 dx = 0 \right\}.$$

Hence, $\text{codim } R(L(b', \theta_a, 0)) = 1$ and $i(w_1, \chi_1) \notin R(L(b', \theta_a, 0))$. The proof of the remainder part is standard, and is omitted here.

Let $L(b(s), u(s), v(s))$ and $L(b, \theta_a, 0)$ be the linearized operators of (1.2) at $(b(s), u(s), v(s))$, $(b, \theta_a, 0)$, respectively. It follows from Lemma 7, Corollary 1.13, and Theorem 1.16 [14] that the following Lemma 8 holds.

LEMMA 8. There exist C^1 -functions: $b \rightarrow (\nu(b), M(b))$ and $s \rightarrow (\pi(s), N(s))$, defined from the neighbourhood of b' and 0 into $R \times X_1$, respectively, such that $\nu(b') = \pi(0) = 0$, $M(b') = N(0) = (w_1, \chi_1)$, and

$$\begin{aligned} L(b, \theta_a, 0)M(b) &= \nu(b)M(b), & \text{for } |b - b'| \ll 1, \\ L(b(s), u(s), v(s))N(s) &= \pi(s)N(s), & \text{for } |s| \ll 1. \end{aligned}$$

Moreover, $\dot{\nu}(b') \neq 0$, whence $\pi(s) \neq 0$, $\pi(s)$ and $-s\dot{b}(s)\dot{\nu}(b')$ have the same sign for $|s| \ll 1$, where $\dot{\nu}(b')$ is the derivative of $\nu(b)$ with respect to b at $b = b'$, and $\dot{b}(s)$ is the derivative of $b(s)$ with respect to s .

LEMMA 9. The derivative of $\nu(b)$ with respect to b at b' is negative.

PROOF. By the Riesz-Schauder theory, the spectrum $\sigma(L(b, \theta_a, 0))$ of $L(b, \theta_a, 0)$ consists of real eigenvalues and

$$\sigma(L(b, \theta_a, 0)) = \sigma(L(a)) \cup \sigma(-\Delta - b - d\theta_a),$$

where $\sigma(L)$ denotes the spectrum of the operator L . Since all the eigenvalues of $L(a)$ are positive, and the least eigenvalue of the operator $-\Delta - b - d\theta_a$ is $(b' - b)$, hence, for $|b - b'| \ll 1$, we have $\nu(b) = b' - b$, which leads to $\dot{\nu}(b') = -1$.

LEMMA 10. The derivative of $b(s)$ with respect to s at $s = 0$ satisfies

$$\dot{b}(0) = \int_{\Omega} \chi_1^2 (\chi_1 - dw_1) dx.$$

PROOF. By substituting $(b(s), u(s), v(s))$ into (1.2), differentiating with respect to s , and then setting $s = 0$, we find that

$$-\Delta \dot{\Psi}(0) = \dot{\Psi}(0) (b' + d\theta_a) + \chi_1 \left(\dot{b}(0) - \chi_1 + dw_1 \right),$$

where $\dot{\Psi}(0)$ is the derivative of Ψ with respect to s at $s = 0$.

Taking the inner product with χ_1 , using Green's formula, and noting the definition of χ_1 , we find

$$\dot{b}(0) = \int_{\Omega} \chi_1^2 (\chi_1 - dw_1) dx.$$

In conclusion, the first part of Theorem 4 is given by combination of Lemmas 7–10. The second part of Theorem 4 is proved by the following lemma.

LEMMA 11. $\{b : (b, u, v) \in C\} = [\underline{b}, +\infty)$, where $\underline{b} \in (\lambda_1(-d\theta_{a+c}), b')$.

PROOF. From the proof of Theorem 3, it is easy to see that $\underline{b} \geq \lambda_1(-d\theta_{a+c})$. Next, it suffices to show that if $b = \underline{b}$, system (1.2) has a positive solution. By the definition of \underline{b} , there exists a sequence $\{(b_i, u_i, v_i) \in C : b_i > \underline{b}, i = 1, 2, \dots\}$ such that $b_i \rightarrow \underline{b}$ and (b_i, u_i, v_i) satisfies

$$\begin{aligned} -\Delta u_i &= u_i \left(a - u_i + \frac{cv_i}{\gamma + v_i} \right), & \text{in } \Omega, \\ -\Delta v_i &= v_i (b_i - v_i + du_i), & \text{in } \Omega, \\ u_i &= v_i = 0, & \text{on } \partial\Omega. \end{aligned}$$

Without loss of generality, we may assume that $b_i < b'$ for all i . Since $b' > \lambda_1(-d\theta_{a+c})$ and $\theta_a \leq u_i \leq \theta_{a+c}$, we obtain

$$0 < v_i \leq \theta_{[b', a+c]}, \quad \text{in } \Omega,$$

where $\theta_{[b', a+c]}$ is the unique positive solution of the following problem:

$$-\Delta v - d\theta_{a+c}v = v(b' - v), \quad \text{in } \Omega, \quad v = 0, \quad \text{on } \partial\Omega.$$

By the L^p estimate and the Sobolev embedding theorem, we can show that, subject to a subsequence if necessary, $(u_i, v_i) \rightarrow (\underline{u}, \underline{v})$ in C^1 for some $\underline{u} \geq \theta_a$ and $\underline{v} \geq 0$. Suppose that $\underline{v} \equiv 0$. Set $V_i = v_i / \|v_i\|$. Then (u_i, V_i) satisfies

$$\begin{aligned} -\Delta u_i &= u_i \left(a - u_i + \frac{cv_i}{\gamma + v_i} \right), & \text{in } \Omega, \\ -\Delta V_i &= V_i(b_i - v_i + du_i), & \text{in } \Omega, \\ u_i &= V_i = 0, & \text{on } \partial\Omega. \end{aligned}$$

We can proceed as before to give a contradiction. Hence, we have shown that $\underline{v} \geq 0, \neq 0$, and $(\underline{u}, \underline{v})$ satisfies the following equations weakly:

$$\begin{aligned} -\Delta \underline{u} &= \underline{u} \left(a - \underline{u} + \frac{c\underline{v}}{\gamma + \underline{v}} \right), & \text{in } \Omega, \\ -\Delta \underline{v} &= \underline{v}(\underline{b} - \underline{v} + d\underline{u}), & \text{in } \Omega. \end{aligned}$$

The elliptic regularity and maximum principle lead to $\underline{v} > 0$. Hence, $(\underline{u}, \underline{v})$ is the coexistence state of (1.2) for $b = \underline{b}$. Therefore, $\underline{b} > \lambda_1(-d\theta_{a+c})$.

PROOF OF PROPOSITION 1. Let $W = dw_1 - \chi_1$. Since $b' = \lambda_1(-d\theta_a) < \lambda_1 < a$ and w_1, χ_1 satisfy

$$\begin{aligned} -\Delta w_1 &= aw_1 - 2\theta_a w_1 + \frac{c}{\gamma}\theta_a \chi_1, & \text{in } \Omega, \\ -\Delta \chi_1 &= b'\chi_1 + d\theta_a \chi_1, & \text{in } \Omega, \end{aligned} \tag{4.1}$$

then we find

$$\begin{aligned} -\Delta W &= (adw_1 - b'\chi_1) - 2\theta_a(dw_1 - \chi_1) + \theta_a \chi_1 \left(\frac{cd}{\gamma} - 2 - d \right) \\ &> aW - 2\theta_a W + \theta_a \chi_1 \left(\frac{cd}{\gamma} - 2 - d \right). \end{aligned}$$

By the condition of Proposition 1, we have

$$L(a)W \equiv -\Delta W - (a - 2\theta_a)W > 0, \quad \text{in } \Omega.$$

It follows from the generalized maximum principle that $W > 0$ in Ω , which completes the proof of Proposition 1.

PROOF OF PROPOSITION 2. The comparison theorem leads to $u \leq U$, where U satisfies

$$\begin{aligned} U_t &= \Delta U + U(a + c - U), & \text{in } \Omega \times (0, \infty), \\ U &= 0, & \text{on } \partial\Omega \times (0, \infty), \\ U(x, 0) &= u_0(x), & \text{in } \Omega. \end{aligned}$$

Since $a > \lambda_1$, we have $\lim_{t \rightarrow \infty} U = \theta_{a+c}$. Thus $\limsup_{t \rightarrow \infty} u \leq \theta_{a+c}$.

Suppose that $b < b' = \lambda_1(-d\theta_a)$. If c is sufficiently small, we have $b < \lambda_1(-d\theta_{a+c})$. Then we can choose $\epsilon > 0$ sufficiently small such that $b + \epsilon < \lambda_1(-d\theta_{a+c})$.

For such ϵ , there exists $T_\epsilon > 0$ such that

$$u \leq \theta_{a+c} + \frac{\epsilon}{d}, \quad \text{for } t \geq T_\epsilon.$$

By use of the comparison theorem, we get $v \leq V$, where V satisfies

$$\begin{aligned} V_t &= \Delta V + V(b - V + d\theta_{a+c} + \epsilon), & \text{in } \Omega \times (T_\epsilon, +\infty), \\ V &= 0, & \text{on } \partial\Omega \times (T_\epsilon, +\infty), \\ V(x, T_\epsilon) &= v(x, T_\epsilon), & \text{in } \Omega. \end{aligned}$$

Hence, we find that $\lim_{t \rightarrow \infty} V = 0$, which leads to $\lim_{t \rightarrow \infty} v = 0$. Similar to the proof of Theorem 1, we can obtain that $\lim_{t \rightarrow \infty} u = \theta_a$.

Suppose that $b > b'$. It follows from (4.1) and the Hopf boundary lemma that $\frac{\partial w_1}{\partial n} < 0$, $\frac{\partial \chi_1}{\partial n} < 0$ on $\partial\Omega$. Since $w_1 = (c/\gamma)L^{-1}(a)(\theta_a\chi_1)$, we have $\dot{b}(0) > 0$ if c is sufficiently small, where $\dot{b}(0)$ is defined in Lemma 10. By using Lemmas 7–10 once again, we find that $\pi(s) > 0$ for $0 < s \ll 1$ in this case. Hence, there is a positive constant $\delta \ll 1$ such that the coexistence state, which lies on the branch $\{(b, u, v) \in C : b' < b < b' + \delta\}$, is linearly stable.

Next, suppose that $b \geq b' + \delta$. We first show that $\forall \varepsilon > 0$ small, and there exists $\underline{c} > 0$ such that if $c < \underline{c}$, then

$$\|u - \theta_a\| + \|v - \theta_{[b,a]}\| < \varepsilon, \quad (4.2)$$

where $(\theta_a, \theta_{[b,a]})$ is the positive solution of the following equations:

$$\begin{aligned} -\Delta u &= u(a - u), & \text{in } \Omega, \\ -\Delta v - duv &= v(b - v), & \text{in } \Omega. \end{aligned} \quad (4.3)$$

If the above assertion is false, then there exists $\varepsilon_0 > 0$, $c_i \rightarrow 0$ and positive solutions (u_i, v_i) of (1.2) with $c = c_i$, such that

$$\|u_i - \theta_a\| + \|v_i - \theta_{[b,a]}\| \geq \varepsilon_0. \quad (4.4)$$

We may assume that $c_i < 1$ for all i . Since $\theta_a \leq u_i \leq \theta_{a+c_i}$, it is easy to show that $\lim_{i \rightarrow \infty} u_i = \theta_a$ uniformly. By the monotone method and the uniqueness of $\theta_{[b,a+1]}$, we find that $0 < v_i \leq \theta_{[b,a+1]}$, where $\theta_{[b,a+1]}$ is the unique positive solution of the following equation:

$$-\Delta v - d\theta_{a+1}v = v(b - v), \quad \text{in } \Omega, \quad v = 0, \quad \text{on } \partial\Omega.$$

We can proceed as before to show that, subject to a subsequence if necessary, $(u_i, v_i) \rightarrow (\hat{u}, \hat{v})$ in C^1 . Hence, we have $\hat{u} = \theta_a$ and $\hat{v} \geq 0$. Suppose that $\hat{v} \equiv 0$. Set $V_i = v_i/\|v_i\|$. Then

$$-\Delta V_i = V_i(b - v_i + du_i), \quad \text{in } \Omega, \quad V_i = 0, \quad \text{on } \partial\Omega.$$

We may assume that $V_i \rightarrow V$ in C^1 , $V \geq 0$, and $V \not\equiv 0$. Then V satisfies weakly

$$-\Delta V = V(b + d\theta_a), \quad \text{in } \Omega, \quad V = 0, \quad \text{on } \partial\Omega.$$

It follows from elliptic regularity and the maximum principle that $V \in C^2$ and $V > 0$. Therefore, $b = \lambda_1(-d\theta_a)$, a contradiction. This implies that $\hat{v} \not\equiv 0$. Similar to the process as above, we can show that $\hat{v} > 0$ and $\hat{v} = \theta_{[b,a]}$, which contradicts (4.4). This completes the proof of (4.2).

It is easy to see that as $c \rightarrow 0$, (1.2) is a perturbation of (4.3). Since (4.3) has a unique stable positive solution $(\theta_a, \theta_{[b,a]})$, it follows from the perturbation of linear operator [15] that the coexistence state of (1.2) is linearly stable for $b \geq b' + \delta$ and $c \ll 1$. Furthermore, we can obtain that the coexistence state of (1.2) is unique for $b > b'$ and $c \ll 1$, which lies on the global bifurcation C . In fact, if $b' < b < b' + \delta$ and $c \ll 1$, then the uniqueness of coexistence state of (1.2) is given in Section 3; if $b \geq b' + \delta$ and $c \ll 1$, for technical reasons, it is easy to check that, with minor modification, equation (4.2) can be extended to $c < 0$ but $|c| \ll 1$. Therefore, for $b \geq b' + \delta$ and $|c| \ll 1$, one easily sees by (1.2) and the implicit function theorem that there is a small neighbourhood of $(0, \theta_a, \theta_{[b,a]})$ in $R \times X_1$ such that the positive solution of (1.2) forms a continuous curve, and can be parameterized by c , $|c| \ll 1$. Hence, if we take $c > 0$ and $c \ll 1$, then the uniqueness of coexistence state of (1.2) is given in this case.

REMARK 6. Similar to the process as in the proof of Proposition 2, we can show that if d is sufficiently small, the coexistence state in the neighbourhood of $(b', \theta_a, 0)$ is also linearly stable.

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